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Scattering operators on Fock space: V. The pseudoscalar mesons

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Abstract. The pseudoscalar mesons associated with $SU(3)_{flavour}$ are used to generate a many-particle Fock space. For the eight pseudoscalar mesons the algebra of operators commuting with $SU(3)_{flavour}$, $A_8^{SU(3)}$, is shown to be isomorphic to an infinite-dimensional Lie algebra with a Cartan-Weyl structure discussed previously. When the Fock space is generated by quark-antiquark pairs, the algebra of operators commuting with $SU(3)_{flavour}$ becomes a direct sum of $A_8^{SU(3)}$ and a finite-dimensional Lie algebra. The relationship between elements of the algebra of commuting operators and multiparticle amplitudes is discussed and a model phase operator is chosen to compute some perturbative amplitudes.

1. Introduction

The pseudoscalar mesons can be used as basis elements in a representation space V of the internal symmetry group $SU(3)_{flavour}$. Since the pseudoscalar mesons are bosons, the appropriate many-particle space (all spacetime variables are suppressed) is the symmetric Fock space $\mathscr{S}(V)$, generated by the representation space V of $SU(3)_{flavour}$. An internal symmetry scattering operator S is then a unitary operator on $\mathscr{S}(V)$, invariant with respect to $SU(3)_{flavour}$.

Let $A_V^{SU(3)}$ denote the algebra of operators that commutes with the SU(3) action on $\mathscr{S}(V)$. If the scattering operator is written as $S = e^{i\hat{\eta}}$, where $\hat{\eta}$, the phase operator, is Hermitian and a polynomial in elements of $A_V^{SU(3)}$, then S will automatically be unitary and invariant.

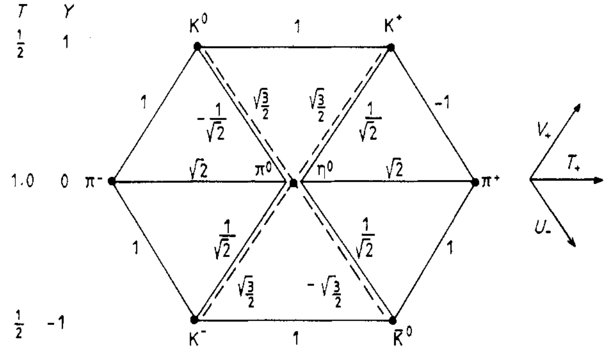
The point of this paper is to investigate $A_V^{SU(3)}$, the algebra of operators commuting with SU(3), for several choices of V. In § 2 we will choose V to be eight-dimensional representation space $V^{(8)}$ of the eight pseudoscalar mesons. It will be shown that $A_8^{SU(3)}$ is isomorphic to the algebra $A_2^{SO(3)}$ studied in Klink (1987, 1988, hereafter referred to as II and IV respectively). Such a result is quite surprising, since not only are the two Fock spaces different, but also the group actions on them are very different. Nevertheless, it will be shown that the number of raising and lowering operators in $A_8^{SU(3)}$ and $A_2^{SO(3)}$ are the same, as are the commutation relations of these operators.

In § 3 we choose V to be a nine-dimensional representation space, generated by quark-antiquark pairs, i.e. the nine pseudoscalar mesons are generated by the tensor product of the two three-dimensional quark-antiquark spaces. The Fock space $\mathscr{S}(V^{(9)}) = \mathscr{S}(V^Q \otimes V^{\bar{Q}})$ will then have an algebra $A_9^{SU(3)}$ of operators commuting with SU(3) that is the direct sum of $A_8^{SU(3)}$ and a finite-dimensional Lie algebra, namely the oscillator algebra.

Finally, using results from IV, we then investigate, in § 4, partial-wave amplitudes for a given phase operator $\hat{\eta}$ and compute some simple multiparticle amplitudes.

2. The algebra $A_8^{SU(3)}$

The octet of pseudoscalar mesons consists of the pairs (K^+, K^0) and $(\bar{K}^0, K^-)$, having isospin $\frac{1}{2}$ and hypercharge 1 and -1, respectively, along with an isotriplet of pions (π^+, π^0, π^-) and an isosinglet η^0, with hypercharge 0. The SU(3) Lie algebra has a basis consisting of three pairs of raising and lowering operators, $T_\pm$, $U_\pm$, $V_\pm$, along with two diagonal operators, T_3, the third component of isospin, and Y, the hypercharge operator (we follow the notation of Gasiorowicz (1966)). The commutation relations of the SU(3) Lie algebra operators are given by Gasiorowicz (1966, p 262). All that is needed to generate the algebra $A_8^{SU(3)}$ are the matrix elements of the raising and lowering operators in the eight-dimensional representation of SU(3). They are also given by Gasiorowicz (1966, p 271) and can be written as



(1)

The dotted lines give the matrix elements of the η^0. For example, $U_-\eta^0 = -(\sqrt{3}/2)\bar{K}^0$ while $V_+\pi^0 = (1/\sqrt{2})K^+$.

The eight-dimensional representation space $V^{(8)}$ generates a many-particle Fock space:

$$\mathscr{S}(V^{(8)}) = \sum_{n=0}^{\infty} \oplus [V^{(8)} \otimes \ldots \otimes V^{(8)}]^n_{\text{sym}}$$

where $[V^{(8)} \otimes \ldots \otimes V^{(8)}]^n_{\text{sym}}$ is the n-fold symmetric tensor product of $V^{(8)}$ with itself. The octet of pseudoscalar mesons forms an orthonormal basis in $V^{(8)}$ and the basis elements are denoted by $|\pi^+\rangle$, $|K^0\rangle, \ldots$. A basis in the many-particle Fock space is then given by products of single-particle basis elements, suitably symmetrised.

As discussed in IV, it is much more convenient to work with holomorphic Hilbert (or Bargmann) spaces than with Fock spaces. Since $V^{(8)}$ is eight dimensional, $\mathscr{S}(V^{(8)}) \approx HL_8^2$, the space of holomorphic functions in eight complex variables. The correspondence between single-particle basis elements in the two spaces is given by

$$\begin{aligned} &|K^+\rangle \to z_{K^+} \\ &|K^0\rangle \to z_{K^0} \\ &\;\vdots \qquad \vdots \end{aligned} \tag{2}$$

Rather than writing z_{K^+} for the complex variable corresponding to the state $|K^+\rangle$, we will just write K^+. Then operations like $D_{K^+}(K^+)^3$ will mean $\partial/\partial z_{K^+}(z_{K^+})^3$, etc.

The action of SU(3) on elements in HL_8^2 is given by

$$(\Gamma_g f)(z) = f(D^{(8)}(g^{-1})z) \qquad f \in HL_8^2,\ g \in \mathrm{SU}(3). \tag{3}$$

$D^{(8)}(g)$ is a matrix element of SU(3) with respect to the basis, equation (1), and

$z \equiv z_{K^+}, \ldots, z_{K^-}$. If g is infinitesimal $D^{(8)}(g)$ becomes the matrix elements given in the weight diagram, equation (1). From this it follows that any operator X in the Lie algebra of SU(3) acts on elements f in HL_8^2 as

$$(Xf)(z) = \sum_{j,j'} C^x_{jj'} z_j \frac{\partial}{\partial z_{j'}} f(z). \tag{4}$$

Here $C^x_{jj'}$ is the matrix element of the operator X as given in equation (1). For example,

$$(V_+f)(z) = \left(\frac{1}{\sqrt{2}} \mathrm{K}^- D_{\pi^0} + \left(\frac{3}{2}\right)^{1/2} \mathrm{K}^- D_{\eta^0} + \bar{\mathrm{K}}^0 D_{\pi^+} \right.$$
$$\left. + \pi^- D_{\mathrm{K}^0} + \frac{1}{\sqrt{2}} \pi^0 D_{\mathrm{K}^+} + \left(\frac{3}{2}\right)^{1/2} \eta^0 D_{\mathrm{K}^+}\right) f(z).$$

As discussed in IV, to construct the algebra of operators that commutes with Γ_g, equation (3), it is necessary to find those polynomials in HL_8^2 that are invariant with respect to Γ_g, and from which all other invariant polynomials can be formed. This is most easily done by examining the tensor product decomposition of n-fold symmetric tensor products of $V^{(8)}$ for small values of N:

n	SU(3) irreps (p, q)									SU(8) irreps	Dim.	
0	(0, 0)									(00 . . . 0)	1	
1		(1, 1)								(10 . . . 0)	8	
2	(0, 0)	(1, 1)	(2, 2)							(20 . . . 0)	36	(5)
3	(0, 0)	(1, 1)	(2, 2)	(30)	(03)	(33)				(30 . . . 0)	120	
4	(0, 0)	2(1, 1)	2(2, 2)			(33)	(40)	(04)	(44)	(40 . . . 0)	330	
⋮										⋮	⋮	

$2(p, q)$ means the representation (p, q) occurs twice. The $(0, 0)$ representation is the identity representation; the dimensions of the other representations are $D(1,1) = 8$, $D(2,2) = 27$, $D(3,3) = 64$, $D(4,4) = 125$, $D(3,0) = D(0,3) = 10$ and $D(4,0) = D(0,4) = 35$. From (5) we see that there will be invariant polynomials for $n = 0$ (which is trivial), $n = 2, 3, 4, \ldots$; they are written as $p^{(m)}(z)$, where (m) is the degree of the polynomial. $p^{(m)}(z)$ can be written as a linear combination of $z_{i_1} \ldots z_{i_m}$, with the linear combination chosen such that all the SU(3) Lie algebra operators, equation (4), annihilate the polynomial. The calculation for the invariant polynomials corresponding to $n = 2$ and 3 gives

$$p^{(2)}(z) = 2\pi^+\pi^- - (\pi^0)^2 - (\eta^0)^2 + 2\mathrm{K}^+\mathrm{K}^- - 2\mathrm{K}^0\bar{\mathrm{K}}^0$$
$$p^{(3)}(z) = 3(\pi^0)^2\eta^0 - 6\pi^+\pi^-\eta^0 - (\eta^0)^3 - 3\sqrt{3}\mathrm{K}^+\mathrm{K}^-\pi^0 - 3\sqrt{3}\mathrm{K}^0\bar{\mathrm{K}}^0\pi^0 \tag{6}$$
$$+ 3K^+K^-\eta^0 - 3\mathrm{K}^0\bar{\mathrm{K}}^0\eta^0 + 3\sqrt{6}\mathrm{K}^0\mathrm{K}^-\pi^+ + 3\sqrt{6}\mathrm{K}^+\bar{\mathrm{K}}^0\pi^-.$$

The fourth-order polynomial $p^{(4)}(z)$ is also readily calculated; it turns out that $p^{(4)}(z) = [p^{(2)}(z)]^2$, so it is not independent.

Though the form of $p^{(2)}(z)$ and $p^{(3)}(z)$ is very different from the two invariant polynomials found in IV, it is striking that there are only two of them and they are of the same degree as those found in IV. This almost guarantees that the algebra $\mathrm{A}_8^{\mathrm{SU(3)}}$ is isomorphic to $\mathrm{A}_2^{\mathrm{SO(3)}}$, for all the other elements come from commutators formed from $p^{(2)}(z)$ and $p^{(3)}(z)$. Note also that the representation structure in (5) is the same as that found in IV, equation (25).

As in IV, raising operators are defined by

$$(Y^{+2}f)(z) \equiv p^{(2)}(z) f(z)$$
$$(Y^{+3}f)(z) \equiv p^{(3)}(z) f(z) \tag{7}$$

while the lowering operators are

$$\begin{aligned}(Y^{-2}f) &\equiv p^{(2)}(D)f(z)\\ (Y^{-3}f)(z) &\equiv p^{(3)}(D)f(z)\end{aligned} \tag{8}$$

where $p^{(m)}(D)$ means replacing the arguments of the polynomial by their corresponding differential operator.

Once these raising and lowering operators have been defined, new invariant operators can be generated by commutators. The most important one is

$$\begin{aligned}6Y^1 &\equiv [Y^{-2}, Y^{+3}]\\ Y^{+1} &= (2\pi^+\eta^0 - \sqrt{6}\mathrm{K}^+\bar{\mathrm{K}}^0)D_{\pi^+} + (2\pi^-\eta^0 - \sqrt{6}\mathrm{K}^-\mathrm{K}^0)D_{\pi^-}\\ &\quad + (2\pi^0\eta^0 - \sqrt{3}\mathrm{K}^0\bar{\mathrm{K}}^0 - \sqrt{3}\mathrm{K}^+\mathrm{K}^-)D_{\pi^0}\\ &\quad + [(\pi^0)^2 - (\eta^0)^2 - \mathrm{K}^0\bar{\mathrm{K}}^0 + \mathrm{K}^+\mathrm{K}^- - 2\pi^+\pi^-]D_{\eta^0}\\ &\quad + (-\mathrm{K}^+\eta^0 + \sqrt{3}\mathrm{K}^+\pi^0 - \sqrt{6}\pi^+\mathrm{K}^0)D_{\mathrm{K}^+}\\ &\quad + (-\mathrm{K}^-\eta^0 + \sqrt{3}\mathrm{K}^-\pi^0 - \sqrt{6}\pi^-\bar{\mathrm{K}}^0)D_{\mathrm{K}^-}\\ &\quad + (-\mathrm{K}^0\eta^0 - \sqrt{3}\mathrm{K}^0\pi^0 + \sqrt{6}\pi^-\mathrm{K}^+)D_{\mathrm{K}^0}\\ &\quad + (-\bar{\mathrm{K}}^0\eta^0 - \sqrt{3}\bar{\mathrm{K}}^0\pi^0 + \sqrt{6}\pi^+\mathrm{K}^-)D_{\bar{\mathrm{K}}^0}\end{aligned} \tag{9}$$

which is a new invariant operator that is not a multiplication operator.

To check that the operators $Y^{\pm1}$, $Y^{\pm2}$ and $Y^{\pm3}$ have the same commutation relations as the corresponding $X^{\pm m}$ operators of IV, the commutators $[Y^{\pm1}, Y^{\pm3}]$ and $[Y^{+2}, Y^{-1}]$ have been explicitly computed, using the concrete realisations of these operators given in equations (7)-(9). Since these computations are long and tedious, the work was actually done with the help of a symbolic manipulation program on a VAX computer. The result is that the commutation relations are the same as those found in IV, and hence the algebra $\mathrm{A}_8^{\mathrm{SU}(3)}$ is isomorphic to $\mathrm{A}_2^{\mathrm{SO}(3)}$, even though the spaces HL_5^2 and HL_8^2 are different and the forms of the polynomial invariants are different.

3. Quarks and the algebra $\mathrm{A}_9^{\mathrm{SU}(3)}$

The algebra of operators commuting with the SU(3) action depends on the representation space of SU(3). In this section we will choose the representation space to be the nine-dimensional tensor product space of quarks and antiquarks. This nine-dimensional space is reducible under SU(3), so the Fock space can be written either as $\mathscr{S}(V^Q \otimes V^{\bar{Q}})$ or $\mathscr{S}(V^{(8)} \oplus V^{(1)})$.

To emphasise the role played by the quarks and antiquarks, the corresponding holomorphic Hilbert space is written $HL_{3\times3}^2$, with z a 3×3 complex matrix whose entries correspond to the nine possible quark-antiquark combinations. The action of SU(3) on $HL_{3\times3}^2$ is given by

$$(\Gamma_g f)(z) = f(g^+ z g) \qquad g \in \mathrm{SU}(3), f \in HL_{3\times3}^2 \tag{10}$$

where g^+ is the adjoint of g. Since $3\otimes\bar{3} = 1\oplus8$, the various quark-antiquark pairs can be written in terms of the eight pseudoscalar mesons and a ninth meson, the η'. The transformations between $3\otimes\bar{3}$ and the eight pseudoscalar mesons are given in Gasiorowicz (1966, p 291), while η' is given by $z_{\eta'} = \mathrm{Tr}\, z$.

From the action of SU(3) on $HL^2_{3\times 3}$, equation (10), it is clear that all the polynomial invariants are of the form

$$p^{(m)}(z)=\mathrm{Tr}\, z^m. \tag{11}$$

If $\mathrm{Tr}\, z=0$, then the two independent invariants are $p^{(2)}(z)=\mathrm{Tr}\, z^2$ and $p^{(3)}(z)=\mathrm{Tr}\, z^3$, which correspond to the two independent invariants of $\mathrm{A}_8^{\mathrm{SU}(3)}$, equation (6).

However, in this section we are interested in finding the algebra of operators commuting with SU(3) on the Fock space generated by quark-antiquark pairs and this means that $\mathrm{Tr}\, z$ is not zero. There is a new first-degree invariant $p^{(1)}(z)=\mathrm{Tr}\, z$ and this will generate a new raising and lowering operator, defined by

$$\begin{aligned}(\tilde{Y}^{+1}f)(z)&=(\mathrm{Tr}\, z)f(z)\\ (\tilde{Y}^{-1}f)(z)&=(\mathrm{Tr}\, D)f(z)\\ &=\left(\frac{\partial}{\partial z_{11}}+\frac{\partial}{\partial z_{22}}+\frac{\partial}{\partial z_{33}}\right)f(z).\end{aligned} \tag{12}$$

There is also a number operator associated with η', namely $\hat{n}_{\eta'}=\mathrm{Tr}\, z[\partial/\partial(\mathrm{Tr}\, z)]$. The commutation relations for these new operators are

$$\begin{aligned}&[\tilde{Y}^{-1},\tilde{Y}^{+1}]=1\\ &[\hat{n}_{\eta'},\tilde{Y}^{\pm 1}]=\pm\tilde{Y}^{\pm 1}\end{aligned} \tag{13}$$

which are the commutation relations of the harmonic oscillator algebra denoted by O (Streater 1967).

Because the η' forms a one-dimensional space, the algebra of operators that commutes with the SU(3) action on the Fock space generated by quark-antiquark pairs is the direct sum of $A_8^{\mathrm{SU}(3)}$ and O. The towers of particles are arranged in the following way:

n	Particles and dimension					SU(9) irrep	Dim.	
0						$(00\ldots0)$	1	
1	$M(8)$	$\eta'(1)$				$(10\ldots0)$	9	
2	$MM(36)$	$M\eta'(8)$	$\eta'\eta'(1)$			$(20\ldots0)$	45	(14)
3	$M^3(120)$	$M^2\eta'(36)$	$M\eta'\eta'(8)$	$(\eta')^3(1)$		$(30\ldots0)$	165	
4	$M^4(330)$	$M^3\eta'(120)$	$M^2(\eta')^2(36)$	$M\eta'^3(8)$	$(\eta')^4(1)$	$(40\ldots0)$	495	
$\vdots$								

where M means the octet of pseudoscalar mesons, the dimensions of the various products are given in parentheses and the dimensions for powers of M are taken from the table, equation (5).

As in § 2 any polynomial in $\mathrm{A}_8^{\mathrm{SU}(3)}\oplus\mathrm{O}$ can now be used to construct a unitary invariant scattering operator. Transitions between the eight pseudoscalar mesons and η' will only occur, however, if the phase operator contains a product of operators from $\mathrm{A}_8^{\mathrm{SU}(3)}$ and O.

More generally, if there are two spaces V_1 and V_2, the Fock space $\mathscr{S}(V_1\oplus V_2)=\mathscr{S}(V_1)\otimes(V_2)$ will have invariant polynomials arising from $(V_1\otimes\ldots\otimes V_1)^n_{\mathrm{sym}}$ (call these $p_1^{(m)}(z)$) and $(V_2\otimes\ldots\otimes V_2)^n_{\mathrm{sym}}$ (call these $p_2^{(m)}(z)$), which generate algebras $\mathrm{A}^{\mathrm{K}}_{V_1}$ and $\mathrm{A}^{\mathrm{K}}_{V_2}$ as discussed in IV. If there are also invariant polynomials coming from tensor products of the form $(V_1\otimes\ldots\otimes V_1)^{n_1}_{\mathrm{sym}}\otimes(V_2\otimes\ldots\otimes V_2)^{n_2}_{\mathrm{sym}}$ (call these $p_{12}^{(m)}(z)$), then the algebra $\mathrm{A}^{\mathrm{K}}_{V_1\oplus V_2}$ will not be just the direct sum of $\mathrm{A}^{\mathrm{K}}_{V_1}$ and $\mathrm{A}^{\mathrm{K}}_{V_2}$, but will have $\mathrm{A}^{\mathrm{K}}_{V_1}$ and $\mathrm{A}^{\mathrm{K}}_{V_2}$ as subalgebras, along with new operators generated from $p_{12}^{(m)}(z)$ that produce transitions between many-particle states from V_1 and V_2. In the example given in this section, namely $V_1 = V^{(8)}$ and V_2‘ = ’η', no mixing was possible because new invariants $p_{12}^{(m)}(z)$ cannot arise from a one-dimensional space.

4. The algebra of commuting operators and scattering amplitudes

To get the connection between the algebra of commuting operators and amplitudes of the ‘scattering’ operator, a representation space V of the internal symmetry group K must be given. V has a basis $\hat{e}_i$, $i = 1, \ldots, N$, of fundamental particles; particles are called fundamental if they are stable under the strong interactions. Note that, under this definition, the vector mesons are not fundamental since they decay strongly into other particles. In this paper we are considering only bosons, so if there are several species of fundamental particles, the relevant Fock space is $\mathscr{S}(V_1)\otimes\mathscr{S}(V_2)\otimes\ldots = \mathscr{S}(V_1\oplus V_2\oplus\ldots)$.

If the Lie algebra of operators commuting with the internal symmetry group on $\mathscr{S}(V_1\oplus V_2\oplus\ldots)$, $\mathrm{A}_V^{\mathrm{K}}$, $V = V_1\oplus V_2\oplus\ldots$, is known, its elements can be used to form the Hermitian phase operator $\hat{\eta}$; then a unitary invariant scattering operator is $\mathrm{e}^{\mathrm{i}\hat{\eta}}$. The goal of this section is to see how to compute amplitudes for a given $\hat{\eta}$, i.e. matrix elements of the form

$$\langle n_1, \ldots, n_N|S|n_1', \ldots, n_N'\rangle$$

the magnitude squared of which gives the probability for a transition occurring from an initial state in which there are n_i' particles of type i, $i = 1, \ldots, N$, to a final state where there are n_i particles of type i.

For the algebra $\mathrm{A}_8^{\mathrm{SU}(3)}$ discussed in § 2, $N = 8$ for the eight pseudoscalar mesons, which are the fundamental particles, and for a given $\hat{\eta}$, the matrix elements of interest are transitions between many-particle states of pseudoscalar mesons.

Since $\hat{\eta}$ is Hermitian, it can be diagonalised. We will write the eigenvectors as $|(\chi)x, \lambda\rangle$, where (χ) is the irreducible representation label for the compact internal symmetry group K, x is a basis label in the representation space (χ) and λ labels the eigenvector in the representation of $\mathrm{A}_V^{\mathrm{K}}$. Then

$$\hat{\eta}|(\chi)x\lambda\rangle = \lambda(\chi)|(\chi)x, \lambda\rangle.$$

Define partial-wave states as states of the form $|(\chi)xn\eta\rangle$, where n is the eigenvalue of the total number operator, and η denotes any other labels arising from $\mathrm{A}_V^{\mathrm{K}}$ needed to uniquely label a state. Then partial-wave amplitudes can be written as

$$\langle(\chi)xn\eta|S|(\chi)xn'\eta'\rangle = \sum_{\lambda}\langle(\chi)xn\eta|(\chi)x\lambda\rangle \mathrm{e}^{\mathrm{i}\lambda(\chi)}\langle(\chi)x\lambda|(\chi)xn'\eta'\rangle \tag{15}$$

where for simplicity we have assumed that the spectrum of $\hat{\eta}$ is discrete. Thus, if the spectrum and the matrix elements of the eigenvectors with respect to partial-wave states of $\hat{\eta}$ are known, the partial-wave amplitudes can be computed. Amplitudes are obtained via Clebsch–Gordan coefficients for many-particle states $\langle n_1, \ldots, n_N|(\chi)xn\eta\rangle$, discussed in IV.

It is, in general, difficult to find the eigenvectors and eigenvalues for a given $\hat{\eta}$. If, however, $\hat{\eta}$ is an element of $\mathrm{A}_V^{\mathrm{K}}$, then partial-wave scattering amplitudes become matrix elements of the group associated with $\mathrm{A}_V^{\mathrm{K}}$, i.e.

$$\langle(\chi)xn\eta|S|(\chi)xn'\eta'\rangle = D^{(\chi)}_{n\eta,n'\eta'}(g) \tag{16}$$

where g is an element of the group associated with A_V^K. This was the point of view taken in II, where A_V^K was a finite-dimensional Lie algebra, and the matrix elements are those of the associated group.

But A_V^K is, in general, an infinite-dimensional Lie algebra. This means, on the one hand, that the possible choices for $\hat{\eta} \in A_V^K$ are much richer than with a finite-dimensional Lie algebra. On the other hand, work is only beginning on the groups associated with infinite-dimensional Lie algebras, such as Kac–Moody algebras (see, for example, Kac 1983); in particular, it is not at all clear what sort of groups are associated with A_V^K.

It is very easy to calculate perturbative amplitudes once $\hat{\eta}$ is given. Choose, for example, $\hat{\eta} = \alpha(Y^{+1} + Y^{-1})$, α real, so that $S = \exp[i\alpha(Y^{+1} + Y^{-1})]$, and write

$$e^{i\hat{\eta}} = \sum \frac{1}{n!} (i)^n (\hat{\eta})^n$$

keeping only the first-order terms; then a typical production reaction matrix element is

$$\begin{aligned} \langle K^+\bar{K}^0\pi^0|S|\pi^+\pi^0\rangle &\approx \langle K^+\bar{K}^0\pi^0|i\alpha(Y^{+1} + Y^{-1})|\pi^+\pi^0\rangle \\ &= i\alpha\langle K^+\bar{K}^0\pi^0|Y^{+1}|\pi^+\pi^0\rangle \\ &= -\frac{i\alpha}{\sqrt{6}} \end{aligned} \tag{17}$$

where use has been made of equation (9); it is clear that such calculations can easily be extended to higher-order terms.

However, a goal of this series of papers is to find non-perturbative methods for calculating strong interaction matrix elements. One such possibility—to be explored in succeeding papers—is to investigate the matrix elements of unitary operators corresponding to group elements associated with the algebra A_V^K. However, it should be pointed out that, if the scattering operator can be written as $S = U_g$, parameters will appear in the group element g which, when sufficiently small, justify a perturbation calculation. In the example given above, if the parameter α is viewed as a group parameter and is sufficiently small, then the perturbation calculation is a good approximation.

5. Conclusion

One method for obtaining non-perturbative scattering amplitudes is to have the phase operator for a quantum system be an element of the Lie algebra of operators that commute with the symmetry operators on the given Hilbert space. Then the scattering operator can be written as the unitary representation operator of some group element of the group associated with the algebra of operators that commute with the symmetry operators; if the group action on the Hilbert space is known, scattering amplitudes will be the matrix elements of the group element which gives the scattering operator.

Such ideas can be applied to internal symmetries. If K is a compact internal symmetry group and V is the representation space of a fundamental set of particles, then the relevant (many-particle) Hilbert space is the Fock space $\mathcal{S}(V)$, and the symmetry action on $\mathcal{S}(V)$ is inherited from the internal symmetry group K. The dual algebra of operators that commutes with K on $\mathcal{S}(V)$ has been denoted by A_V^K and forms the analogue of the SL(2, $\mathbb{R}$) algebra of the example given by Klink (1987).

What we have done in this paper is to compute A_V^K when $K = SU(3)_{flavour}$ and V is either the eight-dimensional representation space of SU(3), in which case $\mathscr{S}(V)$ is the Fock space for the eight pseudoscalar mesons, or V is the nine-dimensional space of quark-antiquark pairs, in which case $\mathscr{S}(V)$ is the Fock space for the nine pseudoscalar mesons. For the eight pseudoscalar mesons the algebra $A_8^{SU(3)}$ turns out to be isomorphic to the algebra of operators analysed in IV, namely $A_2^{SO(3)}$, the algebra of operators commuting with the SO(3) action on the Fock space generated by the five-dimensional representation $l = 2$ of SO(3). Such a result is quite surprising, since the Fock spaces and the symmetry operators acting on these Fock spaces are very different. The fact that the two algebras are isomorphic means that their irreducible representation structure is the same. However, the underlying reason that the structure of the two algebras is the same is not at all clear.

For the Fock space generated by the nine quark-antiquark pairs, the algebra of operators commuting with $SU(3)_{flavour}$ is $A_8^{SU(3)} \times O$, where O is the oscillator algebra (see equation (13)). As discussed in § 3, such a result provides the basis for a more general result, namely whenever the Fock space is generated by a direct sum $V_1 \oplus V_2$, the algebra of operators $A_{V_1 \oplus V_2}^K$ will not be the direct sum of $A_{V_1}^K$ and $A_{V_2}^K$ if there exist invariant polynomials that mix between these two algebras.

A goal of this series of papers is to obtain scattering operators in a non-perturbative way. Thus, a next step is to explore the group structure of the algebra of operators that commute with the given symmetry operators.

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